Multiresolution Analysis

The construction of wavelets from a multiresolution analysis begins by considering the orthogonal complements of V_{m+1} in V_m:

$$W_{m+1} \equiv \left\{ f \in V_m : \left\langle f, g \right\rangle = 0 \text{ for all } g \in V_{m+1} \right\}. \tag{7.22}$$

We write $W_{m+1} = V_m \ominus V_{m+1}$ and $V_m = V_{m+1} \oplus W_{m+1}$. Thus, every $f^m \in V_m$ has a unique decomposition $f^m = f^{m+1} + d^{m+1}$ with $f^{m+1} \in V_{m+1}$ and $d^{m+1} \in W_{m+1}$. Since $W_{m+1} \subset V_m$ and W_m is orthogonal to V_m, it follows that W_{m+1} is also orthogonal to W_m. Similarly, we find that *all the spaces* W_m *(unlike the spaces* V_m*) are mutually orthogonal.* Furthermore, since D^m preserves orthogonality, it follows that

$$W_m = \left\{ D^m f : f \in W_0 \right\} \equiv D^m W_0. \tag{7.23}$$

Let $Q_m : L^2\left(\mathbf{R}\right) \to L^2\left(\mathbf{R}\right)$ denote the orthogonal projection to W_m. Then

$$\begin{aligned} W_m = D^m W_0 &\Rightarrow Q_m = D^m Q_0 D^{-m}, \\ V_m \perp W_m &\Rightarrow P_m Q_m = Q_m P_m = 0, \\ V_m = V_{m+1} \oplus W_{m+1} &\Rightarrow P_m = P_{m+1} + Q_{m+1}. \end{aligned} \tag{7.24}$$

Note also that when $k > m$, then $V_k \subset V_m$ and $W_k \subset V_m$, hence

$$P_k P_m = P_m P_k = P_k, \qquad Q_k P_m = P_m Q_k = Q_k, \qquad k > m. \tag{7.25}$$

The last equation of (7.24) can be iterated by repeatedly replacing the P's with their decompositions:

$$P_m = P_M + \sum_{k=m+1}^{M} Q_k, \qquad M > m \tag{7.26}$$

Any $f \in L^2\left(\mathbf{R}\right)$ can now be decomposed as follows: Starting with the projection $f^m \equiv P_m f \in V_m$ for some $m \in \mathbf{Z}$, we have

$$f^m = P_M f + \sum_{k=m+1}^{M} Q_k f = P_M f^m + \sum_{k=m+1}^{M} Q_k f^m, \tag{7.27}$$

by (7.25). In practice, any signal can only be approximated by a sampled version, which may then be identified with f^m for some *m*. Since the sampling interval for f^m is $\Delta t = 2^m$, we may call 2^{-m} the *resolution* of the signal. Although the "finite" form (7.27) is all that is really ever used, it is important to ask what happens when the resolution of the initial signal becomes higher and higher.

The answer is given by examining the limit of (7.27) as $m \to -\infty$. Since $P_m f \to f$ by (7.21e), (7.27) becomes

$$f = P_M f + \sum_{k=-\infty}^{M} Q_k f, \qquad f \in L^2(\mathbf{R}). \tag{7.28}$$

This gives *f* as a sum of a blurred version $f^M \equiv P_M f$ and successively finer detail $d^k = Q_k f, k \le M$. If the given signal *f* actually belongs to some V_m, then $Q_k f = 0$ for $k \le M$ and (7.28) reduces to (7.27). Equation (7.28) gives an orthogonal decomposition of $L^2(\mathbf{R})$:

$$L^2(\mathbf{R}) = V_M \oplus \bigoplus_{k=-\infty}^{M} W_k. \tag{7.29}$$

If we now let $M \to \infty$ and use (7.21d), we obtain

$$f = \sum_{k=-\infty}^{\infty} Q_k f, \qquad f \in L^2(\mathbf{R}). \tag{7.30}$$

This gives the orthogonal decomposition

$$L^2(\mathbf{R}) = \bigoplus_{k=-\infty}^{\infty} W_k. \tag{7.31}$$

We will see later that the form (7.28) is preferable to (7.30) because the latter only makes sense in $L^2(\mathbf{R})$, whereas the former makes sense in most reasonable spaces of functions. It will be shown that $Q_k f$ is a superposition of wavelets, each of which has zero integral due to the admissibility condition; hence the right-hand size of (7.30) formally integrates to zero, while the left-hand side may not. This apparent paradox is resolved by noting that (7.30) does *not* converge in the sense of $L^1(\mathbf{R})$. No such “paradox” arises from (7.28).